Föreläsning 13/11-13

<u>Markov chains (Markov random processes)</u>

Discrete time process $(X_n;\ n \geq 0)$, $n \in N$

Markov property: $P(\underbrace{X_n = s}_{\text{future}} | \underbrace{X_{n-1} = x_{n-1}}_{\text{now}}, \underbrace{\dots, X_0 = x_0}_{\text{history}}) = P(X_n = s | X_{n-1} = x_{n-1})$

$\iff P(X_n = s | X_{n_1} = x_1, \dots, X_{n_k} = x_k) = P(X_n = s | X_{n_k} = x_k)$ for $0 \leq n_1 \leq \dots \leq n_k < n$

We will without loss of generality always (more or less) assume that Markov process values are among the integers

Transition probability: $p_{ij} = P(X_n = j | X_{n-1} = i) = P(X_n = j | X_{n-1} = i, \dots, X_0 = ..)$

We assume always time homogeneity which means that p_{ij} do not depend on n.

Transition matrix: $P = (p_{ij})_{ij}$

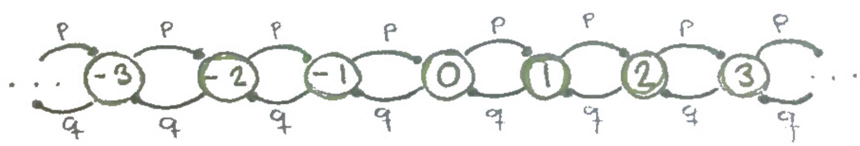
$$\begin{pmatrix} & & \vdots & & \\ & p_{0-1} & p_{00} & p_{01} & p_{02} & \\ \cdots & p_{1-1} & p_{10} & p_{11} & p_{12} & \cdots \\ & p_{2-1} & p_{20} & p_{21} & p_{22} & \\ & & \vdots & & \end{pmatrix}$$

example

Simple random walk

$X_n = \sum_{i=1}^{n} Y_i$ where $Y_1, Y_2, \dots$ are IID r.v.'s with $\begin{cases} P(Y_i = -1) = q = 1-p \\ P(Y_i = 1) = p \end{cases}$

$$P = \begin{array}{c|ccccccc} & & -2 & -1 & 0 & 1 & 2 & \\ \hline 2 & q & 0 & p & & & & \\ 1 & & q & 0 & p & & & \\ 0 & & & q & 0 & p & & \\ -1 & & & & q & 0 & p & \\ -2 & & & & & q & 0 & p \end{array} \iff p_{ij} = \begin{cases} p \text{ for } j = i+1 \\ q \text{ for } j = i-1 \end{cases}$$



m-step transition matrix: $P^{(m)} = (p_{ij}^{(m)})$

made up of m-step transition probabilities.

$p_{ij}^{(m)} = P(X_{n+m} = j | X_n = i) = P(X_{n+m} = j | X_n = i, X_{n-1} = .., \dots, X_0 = ..)$

$\mu^{(n)} = (\dots P(X_n = -1)\ P(X = 0)\ P(X_n = 1) \dots)$ row matrix with matrix elements $\mu_i^{(n)} = P(X_n = i)$

Thm $P^{(m)} = P^m$ and $\mu^{(n)} = \mu^{(0)} P^{(n)} = \mu^{(0)} P^n$

Proof $(P^{(m)})_{ij} = P(X_{n+m} = j \mid X_n = i) = \frac{P(X_{n+m}=j, X_n = i)}{P(X_n = i)} =$

$$= \sum_{\text{all } k} \frac{P(X_{n+m}=j, X_{n+m-1}=k, X_n=i)}{P(X_n=i)} = \sum_{\text{all } k} \frac{P(X_{n+m}=j, X_{n+m-1}=k, X_n=i)}{P(X_{n+m-1}=k, X_n=i)} \cdot$$

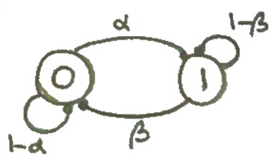
$$\cdot \frac{P(X_{n+m-1}=k, X_n=i)}{P(X_n=i)} = \sum_{\text{all } k} P(X_{n+m}=j \mid X_{n+m-1}=k, \cancel{X_n=i}) P(X_{m+n-1}=k \mid X_n=i)$$

$$= \sum_{\text{all } k} P_{kj} P_{ik}^{(m-1)} = (P^{(m-1)}P)_{ij} \implies P^{(m)} = \underbrace{P^{(m-1)}}_{=P^{(m-2)}P} P = P^{(m-2)}P^2 = \ldots = P^m$$

$$\mu_i^{(n)} = P(X_n = i) = \sum_{\text{all } k} P(X_n = i \mid X_0 = k) P(X_0 = k) = \sum_{\text{all } k} P_{ki}^{(n)} \mu_k^{(0)} = (\mu^{(0)} P^{(n)})_i$$

■

example



$\iff P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$

$(X_n \,;\, n \geq 0)$

$P^n = ?$, Diagonalize $P = Q^{-1} \overset{\text{diagonal matrix}}{D} Q \implies P^n = Q^{-1} \underbrace{DQ \ldots Q^{-1}D}_{n \text{ times}} Q = Q^{-1} D^n Q$

6.2 Classification of states

State (= value) i is persistent (= recurrent) if
$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1$

State i is transitient if $P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) < 1$

Thm i recurrent $\iff \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$, $\quad P_{ij}^{(n)} = P(X_{n+m} = j \mid X_m = i)$

$P_{ii}^{(0)} = 1$ by def. $\quad P_{ii}^{(n)} = P(X_{n+m} = i \mid X_m = i)$

Proof $P_i(z) = \sum_{n=0}^{\infty} P_{ii}^{(n)} z^n \qquad F_i(z) = \sum_{n=0}^{\infty} f_{ii}^{(n)} z^n$

where $f_{ii}^{(n)} = P(X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i \mid X_0 = i)$, $n \geq 1$, $f_{ii}^{(0)} = 0$

$$P_{ii}^{(n)} = P(X_n = i \mid X_0 = i) = \sum_{k=0}^{n} P(X_n = i, \ldots, X_{n-k} = i, X_{n-k-1} \neq i, \ldots, X_1 \neq i \mid X_0 = i)$$

$$= \sum_{k=0}^{n} \frac{P(X_n = i, \ldots, X_{n-k} = i, X_{n-k-1} \neq i, \ldots, X_1 \neq i, X_0 = i)}{P(X_{n-k} = i, X_{n-k-1} \neq i, \ldots, X_1 \neq i, X_0 = i)} \cdot \quad \xrightarrow{\text{forts.}}$$

$$\cdot \; \frac{P(X_{n-k}=i, X_{n-k-1}\neq i, \dots, X_1\neq i, X_0=i)}{P(X_0=i)} =$$

$$= \sum_{k=0}^{n} \underbrace{P(X_n=i \mid X_{n-k}=i, \cancel{X_{n-k-1}\neq i, \dots, X_1\neq i, X_0=i})}_{p_{ii}^{(k)}} f_{ii}^{(n-k)}$$

$$P_i(z) = \sum_{n=0}^{\infty} p_{ii}^{(n)} z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} p_{ii}^{(k)} f_{ii}^{(n-k)} z^n = \{n=0 \Rightarrow p_{ii}^{(0)}=1\} =$$

$$= \underbrace{1}_{p_{ii}^{(0)}} + \sum_{n=1}^{\infty}\sum_{k=0}^{n} p_{ii}^{(k)} f_{ii}^{(n-k)} z^n = 1 + \underbrace{\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} f_{ii}^{(n-k)} z^{n-k}\right) p_{ii}^{(k)} z^k}_{F_i(z)\;\; P_i(z)} \Rightarrow$$

$$F_i(z) = \frac{P_i(z)-1}{P_i(z)}$$

$$P(X_n=i \text{ some } n\geq 1 \mid X_0=i) = \sum_{n=0}^{\infty} f_{ii}^{(n)} = \lim_{z\to 1} F_i(z) = \lim_{z\to 1} \frac{P_i(z)-1}{P_i(z)} =$$

$$= \frac{\sum_{n=0}^{\infty} p_{ii}^{(n)} - 1}{\sum_{n=0}^{\infty} p_{ii}^{(n)}} = 1 \begin{cases} \text{not if } \sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty \\ \text{yes if } \sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty \end{cases}$$